# ON REALIZATION OF TERMINAL CAPACITY MATRICES

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# THESIS

ON REALIZATION OF TERMINAL
CAPACITY MATRICES

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December 1971

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Capacity Matrices

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Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN ELECTRICAL ENGINEERING

from the

NAVAL POSTGRADUATE SCHOOL
December 1971

## ABSTRACT

This paper presents three algorithms for minimum cost synthesis of an oriented communication net. The realization technique is developed using the min-cut max-flow theorem. The algorithms are able to handle higher order terminal capacities compared to previous methods. Necessary and sufficient conditions are given for the application of the algorithms, which are suitable for computer implementation.

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ACKNOWLEDGEMENT

The author wishes to express his appreciation to Professor Shu-Gar Chan for the invaluable aid and councel which he has offered during the preparation of this thesis.

## I. INTRODUCTION

The application of the theory of graphs to the analysis of communication nets is natural in the sense that one may consider the various stations of a communication net as vertices and the channels of communication net as branches (lines drawn between these vertices). Every branch has associated with it a nonnegative number called the branch capacity which indicates the maximum amount of information that can pass through the branch. A communication net must have large enough branch capacities such that all message requirements can reach their destinations simultaneously.

In many practical applications, the maximum allowable communication from station i to station j and the maximum allowable communication from station j to station i may be different. For representing such a system, oriented branches must be used, resulting in an oriented graph. Therefore, the branch capacity matrix and the terminal capacity matrix become assymmetrical.

The purpose of this paper is to investigate a synthesis method for oriented communication nets. The necessary conditions and a realization method for up to three-by-three matrix are given by Tang and Chien [2]. The necessary conditions and realization methods for four-by-four matrix are presented in this paper. These ideas may easily be extended to higher-order cases. The method given here is based on the max-flow min-cut theorem [7] and can be adapted for computer solution. Related flow chart for computer programing will be given later in this paper.

## II. THEORETICAL DEVELOPMENT

Several authors have worked on communication nets and terminal capacity matrices. Methods for the synthesis of oriented or nonoriented communication nets are given in references [2, 3, 5, 6, 7, 8, 10, 12]. Properties of the terminal capacity matrix, the max-flow min-cut theorem and several methods for analyzing communication nets are presented in this section.

### A. PROPERTIES OF TERMINAL CAPACITY MATRIX

#### 1. Oriented Communication Net

The terminal capacity matrix is always partitionable into submatrices and submatrices on the diagonal are again partitionable until each submatrix becomes a one-by-one matrix.

THEOREM 1 [1]. Partitioning of a terminal capacity matrix, if $t_1$ corresponds to a minimum cut $S_1$ cutting all directed paths from subgraph A to subgraph B, and if $t_2$ corresponds to another minimum cut $S_2$ cutting all directed paths from $A_1$ to $A_2$ (both subgraph of A), then $S_2$ cannot be a minimum cut of any two subgraphs of B unless $t_2=t_1$ and if $S_2$ is also a minimum cut cutting all directed paths from some $B_3$ to $B_4$ (both nonempty subgraphs of B), then there exist at least two more cuts with the same minimum value t.

THEOREM 2 [1]. Let $t_{ij}(i,j=1, 2, \ldots\ldots, n, i\neq j)$ be any element of a terminal capacity matrix; then

$$t_{ij} \geq \min(t_{ik}, t_{kj}) \tag{2-1}$$

and $i,j,k=1,2,\ldots\ldots, n, i\neq j$

THEOREM 3 [1]. $T'$ is the terminal capacity matrix of graph $G'$ and $T''$ is the terminal capacity matrix of graph $G''$. Let,

$$G = G' + G'' \text{ (in terms of edge matrices)} \tag{2-2}$$

and

$$T = T' + T'' \tag{2-3}$$

Then $T$ is the terminal capacity matrix of graph $G$ if and only if for each ordered node pair $i$ and $j$ there exists a cut for all three graphs $G$, $G'$ and $G''$.

2. <u>Nonoriented Communication Net</u>

A terminal capacity matrix of a communication net is always partitionable into the submatrices as in oriented case.

The maximal-flow capacity from node $i$ to node $j$ is equal to the maximal-flow capacity from node $j$ to node $i$.

COROLLARY 1 [1]. Let $S$ be the minimum cut-set which separates graph $G$ into subgraph $G'$ and $G''$, the terminal-capacity $t$ is not changed when all edges in $G''$ are shorted, provided that $i$ and $j$ are both in $G'$.

## B. THE MAX-FLOW MIN-CUT THEOREM

The max-flow min-cut theorem is formulated by Ford and Fulkerson [7]. It can be used to obtain maximum flow in a network.

THEOREM 4 [7]. For an oriented network the maximal-flow from node $n_1$ to node $n_2$ is equal to the minimum cut, which cuts all directed paths from $n_1$ to $n_2$.

For finding maximal flow of an oriented network, the following procedure may be used with the aid of the theorem given above.

a) Select a pair of vertices. Determine a path such that all forward edges are not saturated ($f<c$) and all reverse edges have nonzero flow. Repeat it if $f = c$.

b) Let $\Delta f_1$ be the minimum of all the differences $(c-f)$ for forward edges and $\Delta f_2$ be the minimum of all the differences for reverse edges. Increase the flow of the forward edges by an amount $\Delta f = \min(\Delta f_1, \Delta f_2)$, and decrease the flow of reverse edges by an amount $\Delta f$.

c) Repeat (a) and (b) until no more paths exist as described in step (a).

## C. SYNTHESIS OF NONORIENTED COMMUNICATION NETS

Several authors have investigated methods for realizing nonoriented communication nets. In this section, the method of Mayeda [3], Wing and Chien [5], Gomory and Hu [6] will be briefly presented.

Mayeda's method is based on the realization of communication nets using a branch capacity matrix which is obtained from a terminal capacity matrix. The realization is accomplished by partitioning the terminal capacity and the branch capacity matrices properly.

Suppose that the terminal capacity matrix of a communication net is partitioned as

$$T = \left[\begin{array}{c|c} T_{a1} & T_{(t1)} \\ \hline T'_{(t1)} & T_1 \end{array}\right] \qquad (2\text{-}4)$$

Let $N_1$ and $N_{a1}$ be the subnets corresponding to $T_1$ and $T_{a1}$, respectively.

Partition the branch capacity matrix in the following form

$$C = \left[\begin{array}{c|c} C_{a1} & C_{(t1)} \\ \hline C'_{(t1)} & C_1 \end{array}\right] \qquad (2\text{-}5)$$

It can be seen that the rows and the columns of $C_1$, $C_{a1}$, and $C_{(t_1)}$ have the same arrangement as the rows and the columns of $T_1$, $T_{a1}$, and $T_{(t_1)}$, respectively, the branches whose branch capacities appear in $C_{(t_1)}$ are those which are connected between any vertex in $N_1$ and any vertex in $N_{a1}$. Then, $t_1$ in $T_{(t_1)}$ is equal to the sum of all elements (branch capacities) in $C_{(t_1)}$.

$$t_1 = \text{Sum of all elements in } C_{(t_1)} \tag{2-6}$$

Let a principal partitioning process be applied to the resultant submatrix $T_{a1}$ in Eq. (2-4).

$$T = \left[\begin{array}{cc|c} T_{a2} & T_{(t_2)} & \\ \hline T'_{(t_2)} & T_2 & T_{(t1)} \\ \hline \multicolumn{2}{c|}{T'_{(t1)}} & T_1 \end{array}\right] \tag{2-7}$$

Let the branch capacity matrix in Eq. (2-5) is partitioned as

$$C = \left[\begin{array}{c|c|c} C_{a2} & C_{(t2)} & C_{(t_1)_{a2}} \\ \hline C'_{(t2)} & C_2 & C_{(t1)_2} \\ \hline C'_{(t1)_{a2}} & C'_{(t1)_2} & C_1 \end{array}\right] \tag{2-8}$$

where the rows and columns of $C_{a2}$, $C_2$, and $C_{(t2)}$ in (2-8) have the same arrangement as the rows and columns of $T_{a2}$, $T_2$, and $T_{(t_2)}$ in (2-7), respectively.

$t_2$ is equal to any element in $T_{(t_2)}$ and can be written as

$$t_2 = \text{Sum of elements in } C_{(t_2)} + \min\left(\text{Sum of elements in } C_{(t_1)_{a2}},\ \text{Sum of elements in } C_{(t_1)_2}\right) \tag{2-9}$$

Let $V(C_k)$ be the sum of all elements in the submatrix $C_k$. Then (2-9) can be expressed as

$$t_2 = V(C_{(t_2)}) + \min \left\{ V(C_{(t_1)_{a2}}),\ V(C_{(t_1)_2}) \right\} \tag{2-10}$$

Suppose a principle partitioning process is applied to T in (2-7).

$$T = \left[\begin{array}{c|c|c|c} T_{a3} & T_{(t_3)} & & \\ \hline T'_{(t_3)} & T_3 & T_{(t_2)} & T_{(t_1)} \\ \hline \multicolumn{2}{c|}{T'_{(t_2)}} & T_2 & \\ \hline \multicolumn{3}{c|}{T'_{(t_1)}} & T_1 \end{array}\right] \tag{2-11}$$

Let the branch capacity matrix in (2-8) be partitioned as

$$C = \left[\begin{array}{c|c|c|c} C_{a3} & C_{(t_3)} & C_{(t_2)a3} & C_{(t_1)a3} \\ \hline C'_{(t_3)} & C_3 & C_{(t_2)3} & C_{(t_1)3} \\ \hline C'_{(t_2)a3} & C'_{(t_2)3} & C_2 & C_{(t_1)2} \\ \hline C'_{(t_1)a3} & C'_{(t_1)3} & C'_{(t_1)2} & C_1 \end{array}\right] \tag{2-12}$$

Let $N_1$, $N_2$, $N_3$ and $N_{a3}$ be the subnets consisting of the vertices associated to the rows (and the columns) of $T_1$, $T_2$, $T_3$, and $T_{a3}$ respectively. Also let $N(S_3)_1$ and $N(S_3)_2$ be the subnets obtained from the net N by removing every branch in the corresponding cut set $S_3$ of $t_3$ where $N(S_3)_1$ contains $N_{a3}$ and $N(S_3)_2$ contains $N_3$. The subnets $N_1$ and $N_2$ can be in either $N(S_3)_1$ or $N(S_3)_2$. Hence $N(S_3)_1$ and $N(S_3)_2$ is the one of the following four subnets.

1) $N(S_3)_1$ contains $N_{a3}$ and $N(S_3)_2$ contains $N_1$, $N_2$ and $N_3$.
2) $N(S_3)_1$ contains $N_{a3}$ and $N_1$, and $N(S_3)_2$ contains $N_2$ and $N_3$.

3) $N(S_3)_1$ contains $N_{a3}$, $N_2$, and $N(S_3)_2$ contains $N_1$ and $N_3$.

4) $N(S_3)_1$ contains $N_{a3}$, $N_1$ and $N_2$, and $N(S_3)_2$ contains $N_3$ only.

Thus the corresponding cutset $S_3$ of $t_3$ is one of the following four cutsets.

CASE 1) $S_a$ consists of the branches which are connected between any vertex in $N_{a3}$ and any vertex in one of $N_1$, $N_2$ and $N_3$.

CASE 2) $S_b$ consists of the branches which are connected between any vertex in either $N_{a3}$ or $N_1$ and any vertex in either $N_2$ or $N_3$.

CASE 3) $S_c$ consists of the branches which are connected between any vertex in either $N_{a3}$ or $N_2$ and any vertex in either $N_1$ or $N_3$.

CASE 4) $S_d$ consists of the branches which are connected between any vertex in any one of $N_{a3}$, $N_1$ and $N_2$ and any vertex in $N_3$.

The branch capacities of the branches, which are connected between any vertex in $N(S_3)_1$ and any vertex in $N(S_3)_2$, are the elements in C at the intersection of Set-1 and Set-2. Set-1 is the rows representing the vertices of $N(S_3)_1$ and Set-2 is the columns representing the vertices of $N(S_3)_2$. Therefore, the cutset $S_a$ [mentioned above in CASE 1] is the set of elements of C which are the intersections of the rows of $C_{a3}$ (representing the vertices in $N_{a3}$) and the columns of $C_1$, $C_2$ and $C_3$ (representing the vertices in $N_1$, $N_2$, and $N_3$) in (2-12).

$$V(S_a) = V(C_{(t_3)}) + V(C_{(t_2)a2} + V(C_{(t_1)a3}) \qquad (2\text{-}13)$$

The cutset $S_b$ [mentioned above in CASE 2] is the set of elements of C which are the intersections of the rows of $C_{a3}$, and $C_1$ (representing the vertices in $N_{a3}$ and $N_1$) and the columns of $C_2$ and $C_3$ (representing the vertices in $N_2$ and $N_3$) in (2-12).

$$V(S_b) = V(C_{(t_3)}) + V(C_{(t_2)a3}) + V(C'_{(t_1)_3}) + V(C'_{(t_1)_2}) \qquad (2\text{-}14)$$

Likewise the values of $S_c$ and $S_d$ (mentioned above in CASE 2 and CASE 3) are

$$V(S_c) = V(C_{(t_3)}) + V(C_{(t_1)a3}) + V(C'_{(t_2)_3}) + V(C_{(t_1)_2}) \qquad (2\text{-}15)$$

$$V(S_d) = V(C_{(t_3)}) + V(C'_{(t_2)_3}) + V(C'_{(t_1)_3}) \qquad (2\text{-}16)$$

$t_3$ is the minimum value of $V(S_a)$, $V(S_b)$, $V(S_c)$ and $V(S_d)$, $t_3$ is equal to

$$t_3 = V(C_{(t_3)}) + \min \{V(C_{(t_2)a3}) + V(C_{(t_1)a3}),\ V(C_{(t_2)a3}) + V(C'_{(t_1)_3}) + V(C'_{(t_1)_2}),\ V(C_{(t_1)a3}) + V(C'_{(t_2)_3}) + V(C_{(t_1)_2}),\ V(C'_{(t_2)_3}) + V(C'_{(t_1)_3})\} \qquad (2\text{-}17)$$

From (2-17) $V(C_{(t_3)})$ can be find and also the subgraph $N_3$ can be formed.

We can apply the principal partitioning process to $T_{a3}$ in (2-11), and by continuing the same procedure the branch capacity matrix can be obtained. The number of the steps depends on the order of T.

1. <u>Method of Elementary Matrices</u>

The method of elementary matrices [1], [5] requires a maximum of $\frac{1}{2}n(n-1)$ branches, where n is the number of nodes. Elementary terminal capacity matrix can be put into the following form

$$T = \begin{bmatrix} d & t_1 & t_2 & t_3 & \cdots & t_{n-1} \\ t_1 & d & t_2 & t_3 & \cdots & t_{n-1} \\ t_2 & t_2 & d & t_3 & \cdots & t_{n-1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ t_{n-1} & t_{n-1} & t_{n-1} & t_{n-1} & \cdots & d \end{bmatrix} \qquad (2\text{-}18)$$

where $t_1 \geq t_2 \geq t_3 \geq \cdots\cdots \geq t_{n-1}$

Every elementary terminal capacity matrix is guaranteed to be realizable [3].

Figure 2-1 realizes an elementary terminal capacity matrix of order n with minimum total edge-capacity.

If a terminal capacity matrix T of order n is pártitionable as

$$T = \left[\begin{array}{c|c}
 & \begin{matrix} t_o & t_o & \cdots & t_o \\ t_o & t_o & \cdots & t_o \\ \vdots & \vdots & & \vdots \\ t_o & t_o & \cdots & t_o \end{matrix} \\
T_1 & \\ \hline
\begin{matrix} t_o & t_o & \cdots & t_o \\ t_o & t_o & \cdots & t_o \\ \vdots & \vdots & & \vdots \\ t_o & t_o & \cdots & t_o \end{matrix} & T_2
\end{array}\right] \qquad (2\text{-}19)$$

where $t_o = \min_{i,j} (t_{ij})$ and $T_1$ and $T_2$ are elementary terminal-capacity matrices of order k and n-k respectively, T can be realized by a net as shown in Fig. 2-2. The two "linking" branches a and b can be placed between any two pairs of nodes. If the T matrix is partitionable into $T_1$, $T_2$, .... , $T_p$ elementary terminal-capacity matrices, realization is shown in Fig. 2-3. The number of branches required for this realization is at most 2n-p-2, where p is the number of elementary terminal-capacity matrices in a given T.

Example 1. The realization of following terminal capacity matrix is given in Fig. 2-4.

$$T = \left[\begin{array}{ccc|cc|ccc}
1 & 14 & 12 & 4 & 4 & 4 & 4 & 4 \\
14 & 2 & 12 & 4 & 4 & 4 & 4 & 4 \\
12 & 12 & 3 & 4 & 4 & 4 & 4 & 4 \\ \hline
4 & 4 & 4 & 4 & 10 & 6 & 6 & 6 \\
4 & 4 & 4 & 10 & 5 & 6 & 6 & 6 \\ \cline{4-8}
4 & 4 & 4 & 6 & 6 & 6 & 8 & 6 \\
4 & 4 & 4 & 6 & 6 & 8 & 7 & 6 \\
4 & 4 & 4 & 6 & 6 & 6 & 6 & 8
\end{array}\right] \qquad (2\text{-}20$$

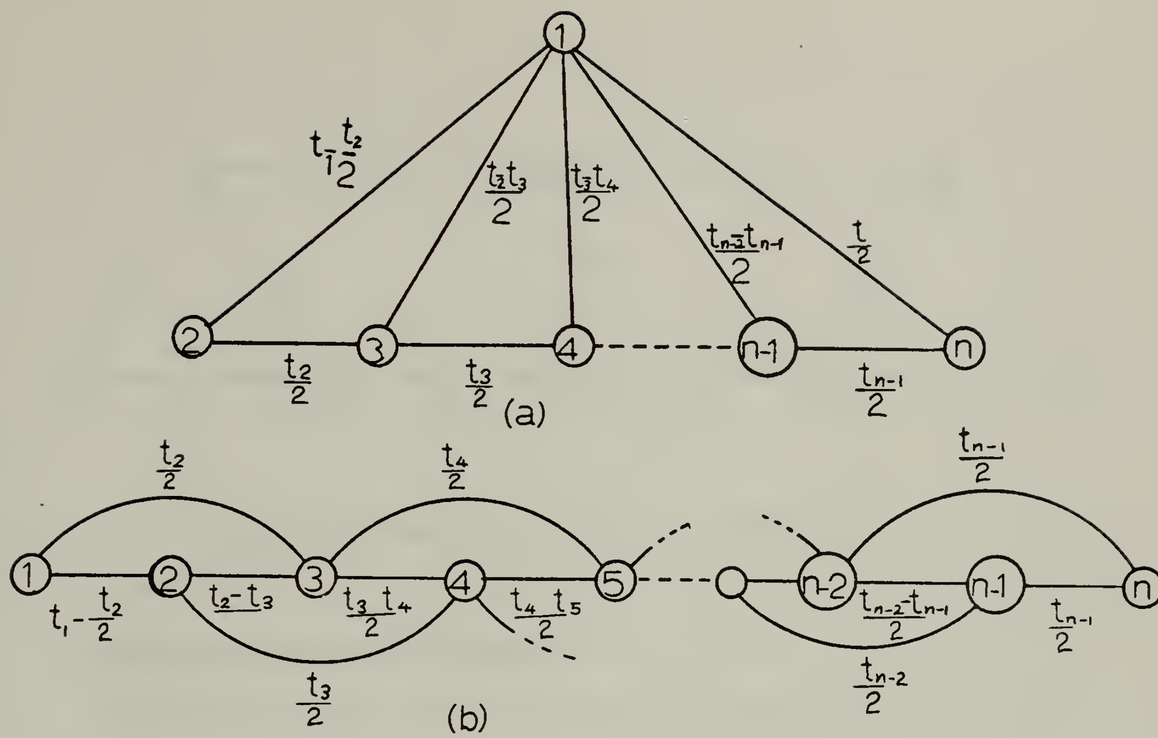



Fig. 2-1 Realization of Elementary Terminal Capacity Matrix

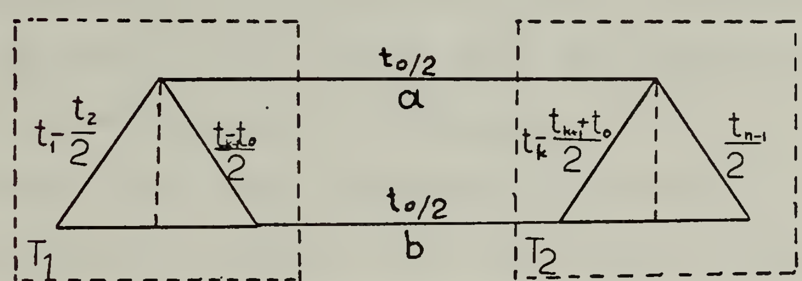



Fig. 2-2 Combination of Elementary Nets

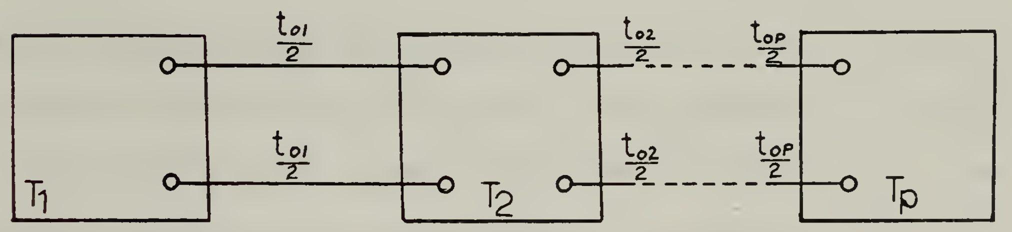



Fig. 2-3 Realization Through Combination of Elementary Nets

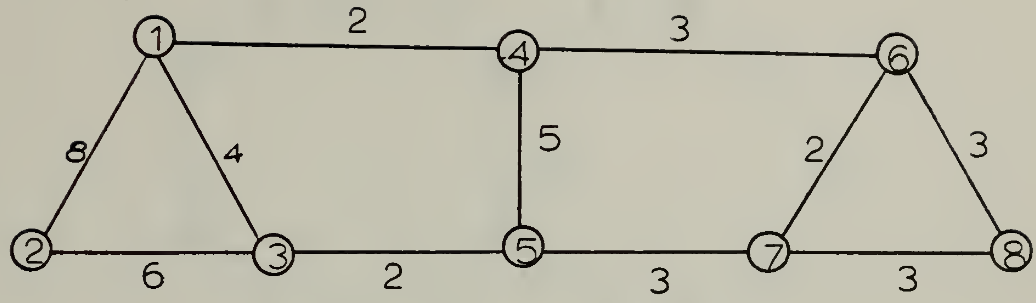



Fig. 2-4 Realization of Eq. (2-20)

2. Method of Successive Expansion

The method of successive expansion [5] employes relatively few edges. The number of edges required by this method is exactly $P'+n-1$, where $P'$ is the index of partitioning and n is the number of nodes. The index of partitioning is the number of operations necessary to partition the matrix T into a form in which every diagonal submatrix is either of order two-by-two or one-by-one.

In this method, first, the diagonal submatrices of the first partition are treated as nodes to form a ring, with each ring element equal to $\frac{1}{2}t_0$, half of the capacity of first partition. To realize each diagonal submatrix, a new ring is formed and each branch in the new ring will have a capacity of $\frac{1}{2}t_1$ except one branch, which has a capacity of $\frac{1}{2}(t_1-t_0)$ and this branch is shared by the new ring and the original ring as shown in Fig. 2-5. Each submatrix is treated as above except one case in which the submatrix is of order two so that the two branches of the new ring may be combined to form one branch.

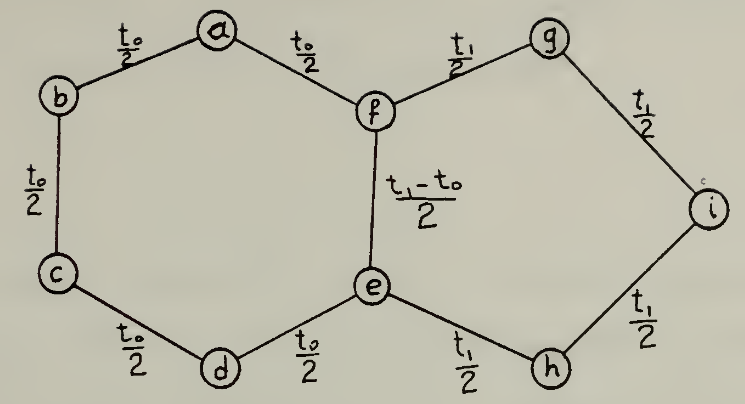



Fig. 2-5 Combination of Ring Structures in the Process of Realization

Example 2. Realization of following terminal capacity matrix is given in Fig. 2-6.

$$
T = \left[\begin{array}{ccc|ccc|cc}
1 & 8 & 6 & 4 & 4 & 4 & 4 & 4 \\
8 & 2 & 6 & 4 & 4 & 4 & 4 & 4 \\
6 & 6 & 3 & 4 & 4 & 4 & 4 & 4 \\
\hline
4 & 4 & 4 & 4 & 10 & 6 & 4 & 4 \\
4 & 4 & 4 & 10 & 5 & 6 & 4 & 4 \\
4 & 4 & 4 & 6 & 6 & 6 & 4 & 4 \\
\hline
4 & 4 & 4 & 4 & 4 & 4 & 7 & 8 \\
4 & 4 & 4 & 4 & 4 & 4 & 8 & 8
\end{array}\right] \tag{2-21}
$$

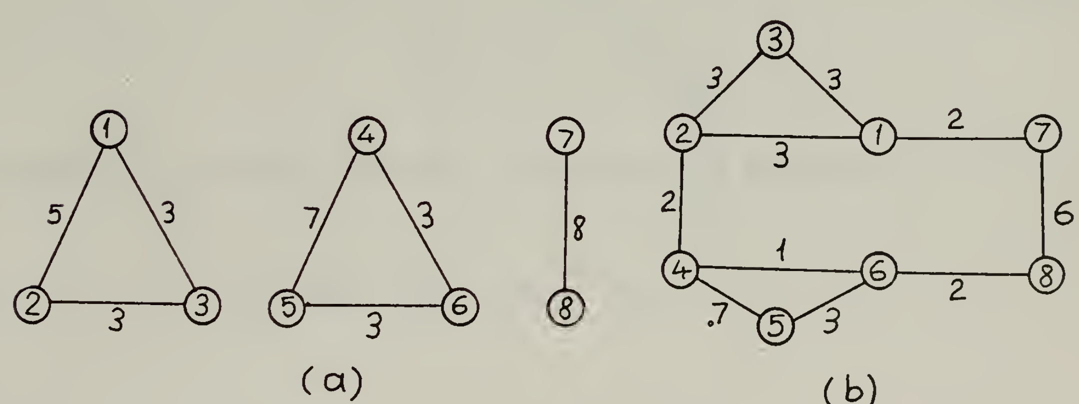


Fig. 2-6 a) Realization of diagonal submatrices.

b) Realization of T matrix.

## 3. Decomposition of Terminal Matrices

With the method of decomposition of terminal matrices [6] the terminal capacity matrix can be written as

$$T = T_1 + T_u \tag{2-22}$$

where $t_u$ is uniform element in $T_u$ and it is equal to the minimum element of $t_{i,j}$. Zero elements of $T_1$ indicate where the minimum cut-set will be in the realization of $T_1$. Minimum cut sets of the realization of $T_u$ correspond to the minimum cut-sets of the realization of $T_1$. T matrix can be written the sum of uniform matrices, as

$$T = \sum T_{u_i} \tag{2-23}$$

The $T_{u_i}$'s are realized and combined in such a way that all their minimum cut-sets correspond to each other.

Example 3. We shall realize the following T matrix with the method described above.

$$T = \begin{bmatrix} 1 & 7 & 6 & 3 & 3 \\ 7 & 2 & 6 & 3 & 3 \\ 6 & 6 & 3 & 3 & 3 \\ 3 & 3 & 3 & 4 & 5 \\ 3 & 3 & 3 & 5 & 5 \end{bmatrix} \tag{2-24}$$

T matrix can be written the sum of four uniform matrices, as

$$T = T_{u_1} + T_{u_2} + T_{u_3} + T_{u_4}$$

$$T_{u_1} = \begin{bmatrix} 1 & 3 & 3 & 3 & 3 \\ 3 & 2 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 4 & 3 \\ 3 & 3 & 3 & 3 & 5 \end{bmatrix} \tag{2-25}$$

$$T_{u_2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 2 & 5 \end{bmatrix} \tag{2-26}$$

$$T_{u_3} = \begin{bmatrix} 1 & 3 & 3 & 0 & 0 \\ 3 & 2 & 3 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \tag{2-27}$$

$$T_{u_4} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \tag{2-28}$$

Realization of each $T_{u_i}$ matrix is given in Fig. 2-7 and realization of the T matrix is given in Fig. 2-8.

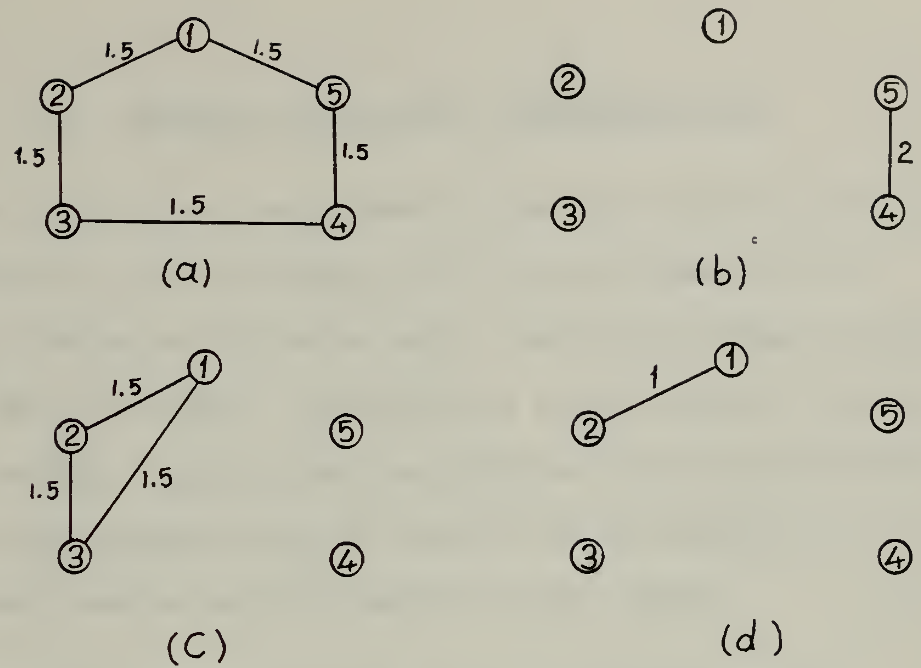



Fig. 2-7 Realization of $T_{u_1}$ in (a), $T_{u_2}$ in (b), $T_{u_3}$ in (c) and $T_{u_4}$ in (d)

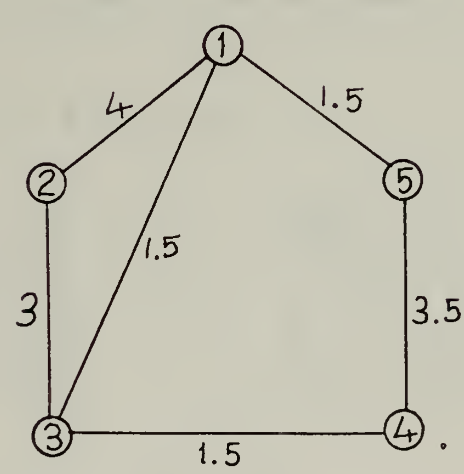



Fig. 2-8. Realization of T matrix

## III. SYNTHESIS OF ORIENTED COMMUNICATION NET

Several methods are investigated by Resh [8], Frisch and Sen [9], Tang and Chien [2], Hu and Gomory [12], Chou and Frank [10] for realizing oriented communication nets. The method of Tang and Chien is given in the next section. It applies to a three-by-three terminal capacity matrix. In this paper the technique for the realization of a four-by-four terminal capacity matrices and its extension to higher-order terminal capacity matrices will be given.

### A. SYNTHESIS OF TERMINAL CAPACITY MATRIX IN THREE-NODE CASE

The terminal capacity matrix is partitioned as

$$T = \left[\begin{array}{c|c|c} (1) & \multicolumn{2}{c}{t_1} \\ \hline t_{21} & (2) & t_{23} \\ \hline t_{31} & t_{32} & (3) \end{array}\right] \qquad (3\text{-}1)$$

It can be written as the sum of two matrices, $T'$ and $T_c$

$$T = T' + T_c$$

$$T' = \left[\begin{array}{c|c|c} (1) & \multicolumn{2}{c}{t_1} \\ \hline t'_{21} & (2) & t'_{23} \\ \hline t'_{31} & t'_{32} & (3) \end{array}\right] \quad ; \quad T_c = \left[\begin{array}{c|c|c} (1) & t_1 & t_1 \\ \hline t_1 & (2) & t_1 \\ \hline t_1 & t_1 & (3) \end{array}\right]$$

where $t'_{ij} = t_{ij} - t_1 \geq 0 \quad$ for $\; i \neq j$

To realize $T'$, we first realize the two-by-two submatrix (containing nodes 1 and 2) as in Fig. 3-1a. Since the first row in $T'$ contains zero entires the only connection between the subgraph shown in Fig. 3-1a and node 1 should be from nodes 2 and 3 to node 1 as shown in Fig. 3-1b with branch capacities x and y. In order to realize $T'$ as in Fig. 3-1b the minimum cut requirements for T must be satisfied. We can obtain the following equations.

$$\min\left[(x+y),\ (x+t'_{23})\right] = t'_{21} \tag{3-2}$$

$$\min\left[(x+y),\ (y+t'_{32})\right] = t'_{31} \tag{3-3}$$

or

$$x = \max\left[(t'_{21}-t'_{23}),(t'_{21}-y)\right] \tag{3-4}$$

$$y = \max\left[(t'_{31}-t'_{32}),(t'_{31}-x)\right] \tag{3-5}$$

If we represent equations (3-4) and (3-5) on the x-y plane, we obtain two curves as shown in Fig. 3-2. The intersection of these curves is $(x_0,y_0)$ where, if $t'_{31} \geq t'_{21}$,

$$x_0 = t'_{21}-t'_{23} \tag{3-6}$$

$$y_0 = \max\left[(t'_{31}-t'_{32}),(t'_{31}-t'_{21}+t'_{23})\right] \tag{3-7}$$

if $t'_{31} \geq t'_{21}$,

$$x_0 = \max\left[(t'_{21}-t'_{23}),(t'_{21}-t'_{31}+t'_{32})\right] \tag{3-8}$$

$$y_0 = t'_{31}-t'_{32} \tag{3-9}$$

The realization of constant matrix $T_c$ is a graph with constant cuts. It is either a cycle oriented in either direction, or two cycles oriented in different directions. Minimum cuts of T, $T'$ and $T_c$ are idential, then conditions are satisfied in Theorem-3 .

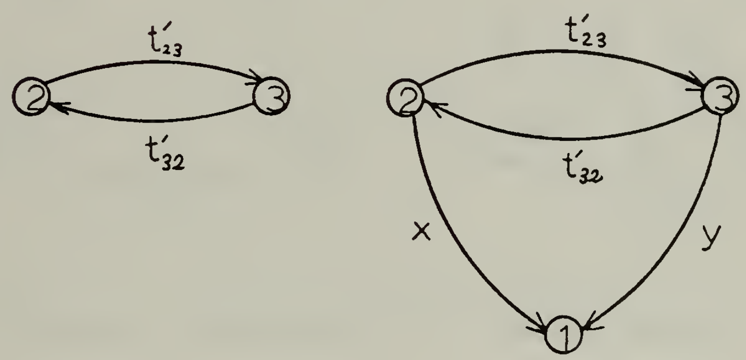



Fig. 3-1. Realization of $T'$

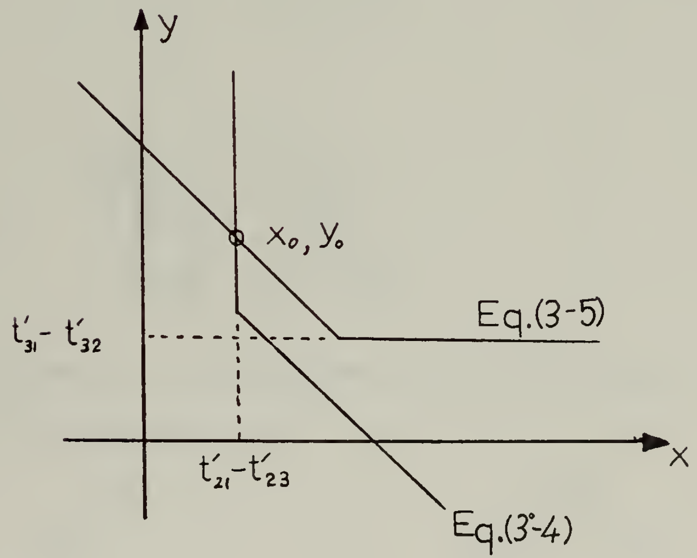



Fig. 3-2. Curves of Equations (3-4) and (3-5)

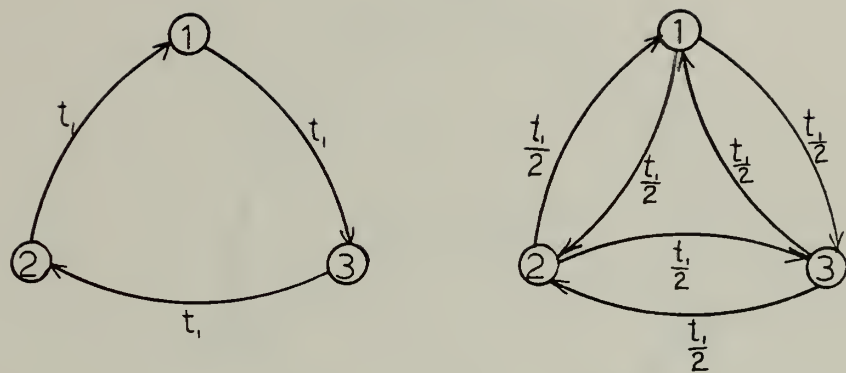



Fig. 3-3. Realization of a Constant Matrix

Example 4. The following T matrix is to be realized:

$$T = \left[\begin{array}{c:cc} 1 & 2 & 2 \\ \hdashline 3 & 2 & 3 \\ 3 & 4 & 3 \end{array}\right] \tag{3-10}$$

We may write:

$$T = T' + T_c = \left[\begin{array}{c:cc} 1 & 0 & 0 \\ \hdashline 1 & 2 & 1 \\ 1 & 2 & 3 \end{array}\right] + \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 3 \end{bmatrix} \tag{3-11}$$

The realizations of $T'$ and $T_c$ are in Fig. 3-4a and b. The final realization of T is in Fig. 3-5.


Fig. 3-4. Realizations of $T'$ and $T_c$ matrices

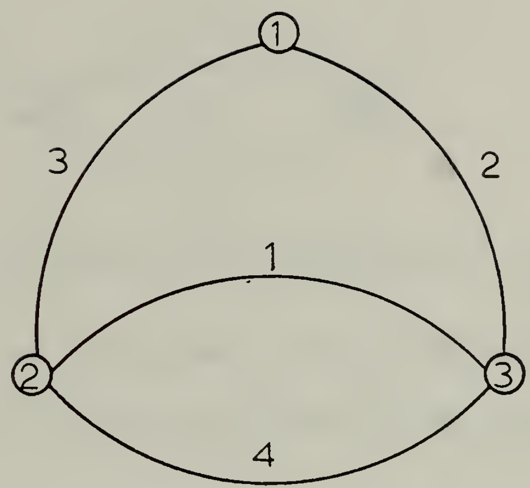



Fig. 3-5. The final realization of T matrix

B. SYSTHESIS OF TERMINAL CAPACITY MATRIX IN FOUR-NODE CASE

The terminal capacity matrix of an oriented communication net containing four nodes can be partitioned as

$$T = \left[\begin{array}{cc|cc} 1 & t_{12} & t_1 & t_1 \\ t_{21} & 2 & t_1 & t_1 \\ \hline t_{31} & t_{32} & 3 & t_{34} \\ t_{41} & t_{42} & t_{43} & 4 \end{array}\right] \qquad (3\text{-}12)$$

The form of the realization is given in Fig. 3-6, where $t'_{ij}=t_{ij}-t_1$.

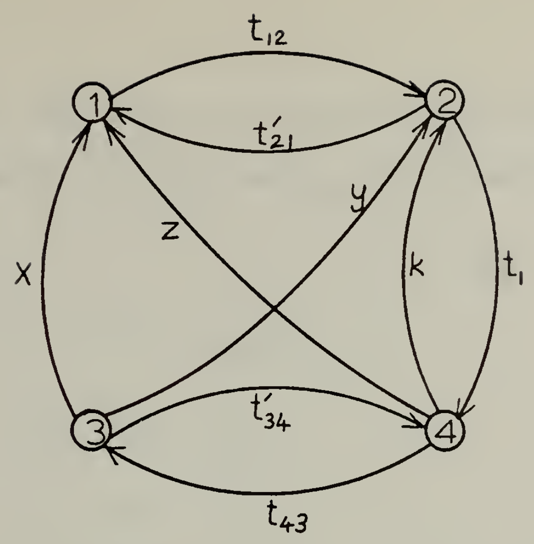



Fig. 3-6. The form of the realization of T matrix

In order to use the algorithms which will be given later in this section, the terminal capacity matrix must be in one of the following three forms.

Form (1) $t_{31} < t_{41}$ ; $t_{31} < t_{42}$

$t_{32} < t_{41}$ ; $t_{32} < t_{42}$

Form (2) $t_{31} < t_{32}$ ; $t_{31} < t_{42}$

$t_{41} < t_{32}$ ; $t_{41} < t_{42}$

Form (3) $t_{31} < t_{32}$ ; $t_{31} < t_{41}$

$t_{42} < t_{32}$ ; $t_{42} < t_{41}$

Form (1) can be realized with Algorithm-A to be given later in section 3-B-1. Form (2) and Form (3) can be realized with Algorithm-B and Algorithm-C to be given later in sections 3-B-2 and 3-B-3, respectively. If the terminal capacity matrix is not in one of these forms, its rows and columns may be rearranged and put into the form of one of them.

The following conditions are necessary for the realization of a four-by-four terminal capacity matrix, using the algorithms given later.

$$t_{31} \geq t'_{21} + t'_{34} + t_1$$

$$t_{32} \geq t_{12} + t'_{34}$$

$$t_{41} \geq t_{43} + t'_{21}$$

$$t_{42} \geq t_{43} + t_{12}$$

1. Algorithm A

1) Determine x and y by writing the following equations, with the aid of Fig. 3-7:

$$t_{31} = \min \left[ (x+y+t'_{34}),\ (x+t'_{21}+t'_{34}+t_1) \right] \tag{3-13}$$

$$t_{32} = \min \left[ (x+y+t'_{34}),\ (y+t_{12}+t'_{34}) \right] \tag{3-14}$$

Which we obtain

$$x = \max \left[ (-y+t_{31}-t'_{34}),\ (t_{31}-t'_{21}-t'_{34}-t_1) \right] \tag{3-15}$$

$$y = \max \left[ (-x+t_{32}-t'_{34}),\ (t_{32}-t_{12}-t'_{34}) \right] \tag{3-16}$$

2) There are several sets of equations for determining z and k, choose proper case, use related equations and apply to Fig. 3-8 for obtaining z and k.

CASE 1: $x+y \leq t_{43}$

$$t_{41} = \min \left[ (x+y+z+k),\ (x+z+t'_{21}) \right] \tag{3-17}$$

$$t_{42} = \min \left[ (x+y+z+k),\ (y+k+t_{12}) \right] \tag{3-18}$$

Which we get

$$z = \max\left[(-k+t_{41}-x-y),\ (t_{41}-x-t'_{21})\right] \tag{3-19}$$

$$k = \max\left[(-z+t_{42}-x-y),\ (t_{42}-y-t_{12})\right] \tag{3-20}$$

CASE 2: $x+y > t_{43}$

a. $x \geq t_{43}$

$$t_{41} = \min\left[(z+k+t_{43}),\ (z+t_{43}+t'_{21})\right] \tag{3-21}$$

$$z = \max\left[(-k+t_{41}-t_{43}),\ (t_{41}-t_{43}-t'_{21})\right] \tag{3-22}$$

b. $x < t_{43}$

$$t_{41} = \min\left[(z+k+t_{43}),\ (z+x+t'_{21})\right] \tag{3-23}$$

$$z = \max\left[(-k+t_{41}-t_{43}),\ (t_{41}-x-t'_{21})\right] \tag{3-24}$$

c. $y \geq t_{43}$

$$t_{42} = \min\left[(z+k+t_{43}),\ (k+t_{43}+t_{12})\right] \tag{3-25}$$

$$k = \max\left[(-z+t_{42}-t_{43}),\ (t_{42}-t_{43}-t_{12})\right] \tag{3-26}$$

d. $y < t_{43}$

$$t_{42} = \min\left[(z+k+t_{43}),\ (k+y+t_{12})\right] \tag{3-27}$$

$$k = \max\left[(-z+t_{42}-t_{43}),\ (t_{42}-y-t_{12})\right] \tag{3-28}$$

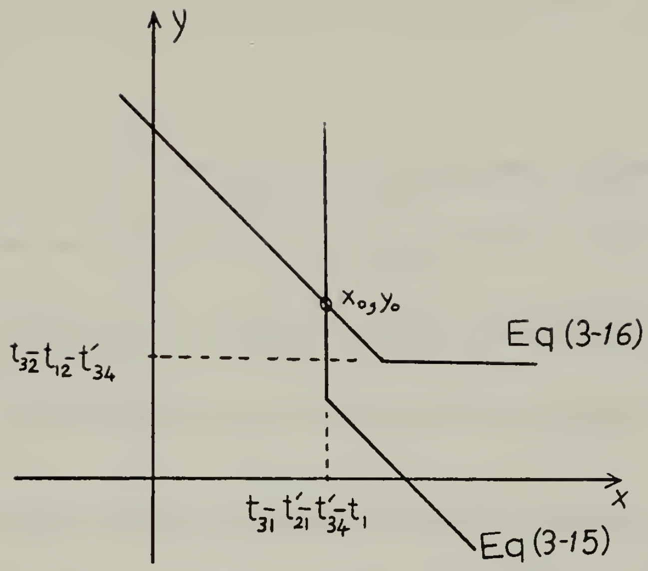



Fig. 3-7. "Curves" of (3-15) and (3-16)

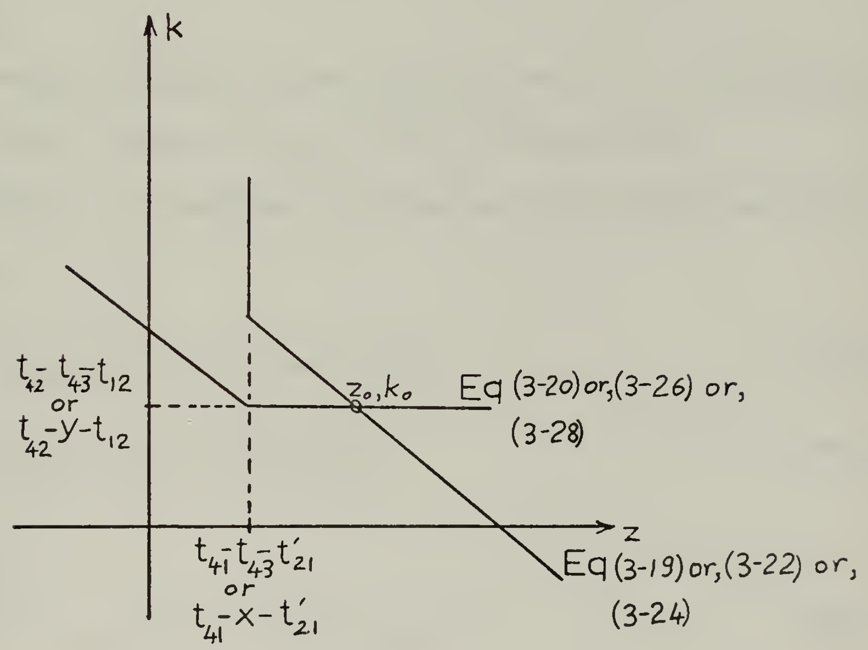



Fig. 3-8. "Curves" of (3-19) or (3-22) or (3-24) and (3-20) or (3-26) or (3-28)

Proof of Algorithm-A: Assume the form of realization as given in Fig. 3-6. Using the max-flow min-cut theorem the following equations can be written.

$$t_{31} = \min\left[(x+y+z+k),(x+z+t'_{21}),(x+y+t'_{34}),(x+t'_{21}+t'_{34}+t_1)\right] \tag{3-29}$$

$$t_{32} = \min\left[(x+y+z+k),(y+k+t_{12}),(x+y+t'_{34}),(y+t_{12}+t'_{34})\right] \tag{3-30}$$

$$t_{41} = \min\left[(x+y+z+k),(z+k+t_{43}),(x+z+t'_{21}),(z+t_{43}+t'_{21})\right] \tag{3-31}$$

$$t_{42} = \min\left[(x+y+z+k),(z+k+t_{43}),(y+k+t_{12}),(k+t_{43}+t_{12})\right] \tag{3-32}$$

Algorithm-A applies under the following cases, $t_{31} < t_{41}$ and $t_{42}$, $t_{32} < t_{41}$ and $t_{42}$. Then, from equations (3-29) through (3-32) we can state that

(a) $x+y+z+k$ can not be min-cut for $t_{31}$ and $t_{32}$ because it is in $t_{41}$ and $t_{42}$.

(b) $x+z+t'_{21}$ and $y+k+t_{12}$ can not be min-cut for $t_{31}$ and $t_{32}$ respectively because they are in $t_{41}$ and $t_{42}$ respectively.

Then, the following two equations are obtained from (3-29) and (3-30) for $t_{31}$ and $t_{32}$:

$$t_{31} = \min\left[(x+y+t'_{34}),(x+t'_{21}+t'_{34}+t_1)\right] \tag{3-33}$$

$$t_{32} = \min\left[(x+y+t'_{34}),(y+t_{12}+t'_{34})\right] \tag{3-34}$$

We obtain

$$x = \max\left[(-y+t_{31}-t'_{34}),(t_{31}-t'_{21}-t'_{34}-t_1)\right] \tag{3-35}$$

$$y = \max\left[(-x+t_{32}-t'_{34}),(t_{32}-t_{12}-t'_{34})\right] \tag{3-36}$$

We can solve the above two equations for x and y or we may use a graphical solution for obtaining x and y.

For the determination of z and k, there are two cases to be considered; (1) $x+y \le t_{43}$ and (2) $x+y > t_{43}$. These cases will give different sets of equations.

Case-1) when $x+y \le t_{43}$, automatically both x and $y \le t_{43}$. So $z+x+t'_{21} \le z+t_{43}+t'_{21}$, then $z+t_{43}+t'_{21}$ is eliminated from (3-31). Also $x+y+z+k \le z+k+t_{43}$ because $t_{43} \ge x+y$, then $z+k+t_{43}$ is eliminated from (3-31) because it can not be min-cut for $t_{41}$. Thus, (3-31) becomes

$$t_{41} = \min\left[(x+y+z+k),(x+z+t'_{21})\right] \tag{3-37}$$

We can apply the same logic to the equation (3-32). $x+y \le t_{43}$ so $y \le t_{43}$, then $z+k+t_{43}$ and $k+t_{43}+t_{12}$ are eliminated from (3-32), where (3-32) becomes

$$t_{42} = \min\left[(x+y+z+k),(y+k+t_{12})\right] \tag{3-38}$$

from which we get

$$z = \max\left[(-k+t_{41}-x-y),(t_{41}-x-t'_{21})\right] \tag{3-39}$$

$$k = \max\left[(-z+t_{42}-x-y),(t_{42}-y-t_{12})\right] \tag{3-40}$$

The solution for x and y is found as before. In order to obtain z and k we can use a graphical solution using (3-39) and (3-40).

Case 2) When $x+y > t_{43}$, we can eliminate $x+y+z+k$ from (3-31) and (3-32) because $x+y+z+k > Z+k+t_{43}$ then it can not be min-cut for $t_{41}$ and $t_{42}$. Thus equations (3-31) and (3-32) become

$$t_{41} = \min\left[(z+k+t_{43}),(z+t_{43}+t'_{21}),(x+z+t'_{21})\right] \tag{3-41}$$

$$t_{42} = \min\left[(z+k+t_{43}),(k+t_{43}+t_{12}),(k+y+t_{12})\right] \tag{3-42}$$

In this case $x \gtrless t_{43}$ and $y \gtrless t_{43}$, and the right-hand sides of equations (3-41) and (3-42) are affected as follows:

$x \geq t_{43}$ eliminates $x+z+t'_{21}$ from (3-41)

$x < t_{43}$ eliminates $z+t_{43}+t'_{21}$ from (3-41)

$y \geq t_{43}$ eliminates $k+y+t_{12}$ from (3-42)

$y < t_{43}$ eliminates $k+t_{43}+t_{12}$ from (3-42)

and each case will give a set of equations for determining z and k.

Example 5. We shall realize the following terminal capacity matrix with using Algorithm-A.

$$T = \left[\begin{array}{cc|cc} 1 & 6 & 3 & 3 \\ 5 & 2 & 3 & 3 \\ \hline 10 & 11 & 3 & 6 \\ 12 & 13 & 7 & 4 \end{array}\right] \tag{3-43}$$

1) Applying the numerical values to (3-15) and (3-16) the following equations are obtained:

$$x = \max\left[(-y+7),(2)\right] \tag{3-44}$$

$$y = \max\left[(-x+8),(2)\right] \tag{3-45}$$

With the above equations and using Fig. 3-9, x=2 and y=6 are obtained.

2) $x+y=8 > t_{43}=7$ then Case-2, also $x < t_{43}=7$ then Case 2b and $y < t_{43}=7$ then Case 2d. Applying the numerical values to (3-24) and (3-28) the following equations are obtained:

$$z = \max\left[(-k+5),(8)\right] \tag{3-46}$$

$$k = \max\left[(-z+6),(1)\right] \tag{3-47}$$

With above equations and using Fig. 3-10, z=8 and k=1 are obtained.

Realization of T is given in Fig. 3-11.

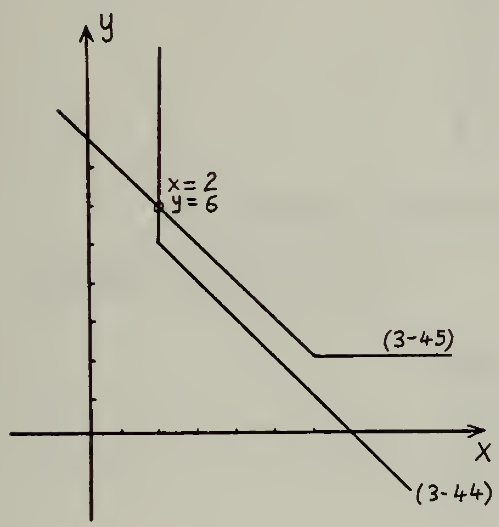



Fig. 3-9

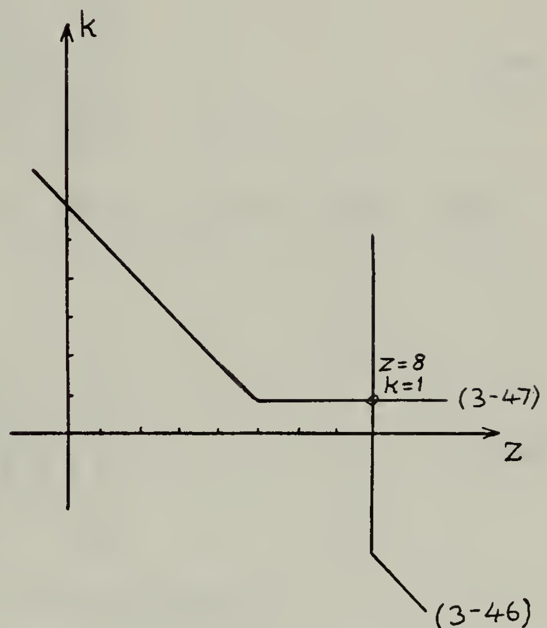



Fig. 3-10

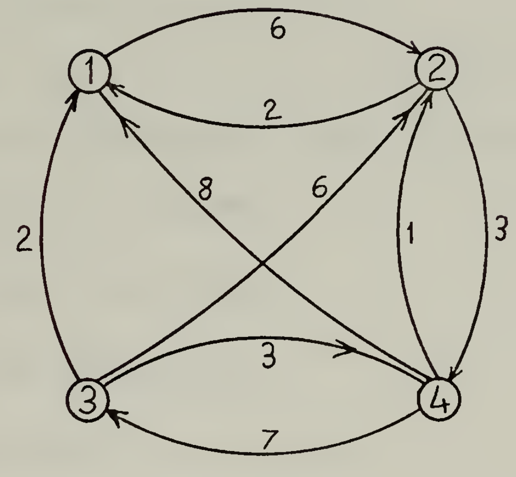



Fig. 3-11. Realization of T matrix in (3-43)

Example 6. We shall realize the following terminal capacity matrix.

$$T = \left[\begin{array}{cc:cc} 1 & 5 & 2 & 2 \\ 4 & 2 & 2 & 2 \\ \hdashline 9 & 11 & 3 & 5 \\ 8 & 8 & 3 & 4 \end{array}\right] \qquad (3\text{-}48)$$

The terminal capacity matrix can be put into Form-1 with changing row 3 by row 4.

$$T = \left[\begin{array}{cc:cc} 1 & 5 & 2 & 2 \\ 4 & 2 & 2 & 2 \\ \hdashline 8 & 8 & 3 & 3 \\ 9 & 11 & 5 & 4 \end{array}\right] \qquad (3\text{-}49)$$

Realization of (3-49) can be done with Algorithm-A.

1) $x = \max\left[(-y+7),(3)\right]$ (3-50)

$y = \max\left[(-x+7),(2)\right]$ (3-51)

x=4 and y=3 are obtained from Fig. 3-12.

2) $x+y=7 > t_{43}=5$ and $x < t_{43}$ then Case-2b, $y < t_{43}$ then Case-2d. Applying the numerical values to (3-24) and (3-28) the following equations are obtained.

$z = \max\left[(-k+4),(3)\right]$ (3-52)

$k = \max\left[(-z+6),(3)\right]$ (3-53)

z=3 and k=4 are obtained from Fig. 3-13.

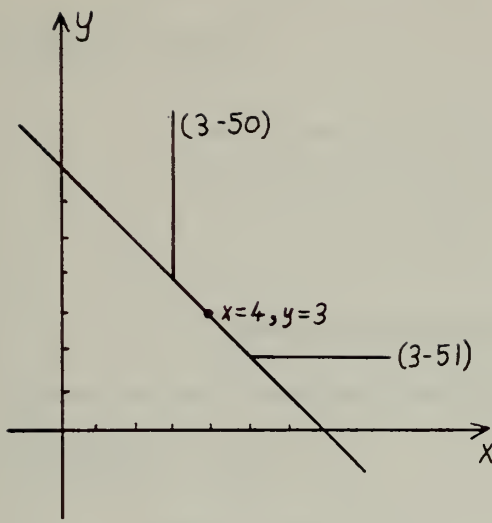



Fig. 3-12

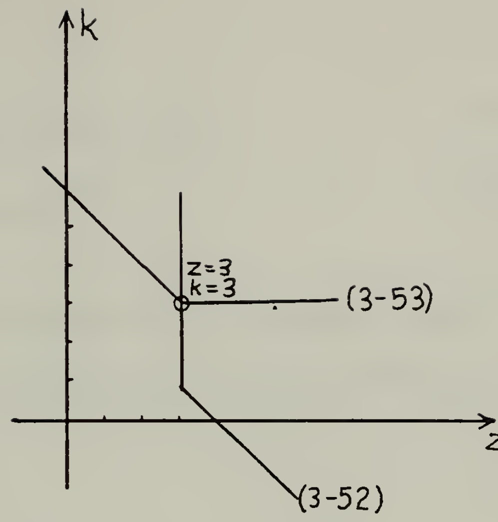



Fig. 3-13

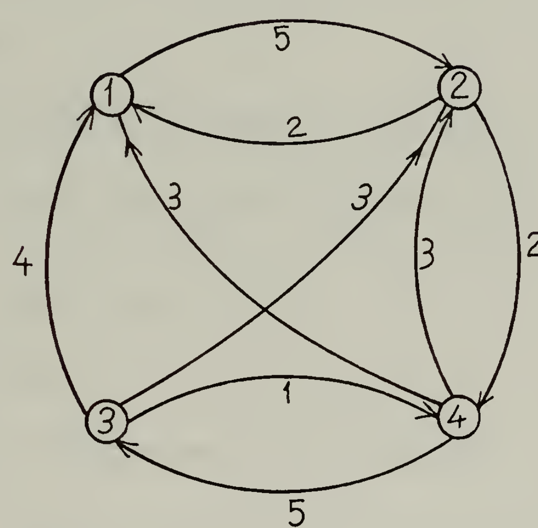



Fig. 3-14. Realization of T matrix in (3-48)

2. Algorithm-B

1) Determine x and z by writing the following equations with the aid of Fig. 3-15:

$$t_{31} = \min\left[(x+z+t'_{21}),\ (x+t'_{21}+t'_{34}+t_1)\right] \tag{3-54}$$

$$t_{41} = \min\left[(x+z+t'_{21}),\ (z+t_{43}+t'_{21})\right] \tag{3-54}$$

Which we obtain

$$x = \max\left[(-z+t_{31}-t'_{21}),\ (t_{31}-t'_{21}-t'_{34}-t_1)\right] \tag{3-55}$$

$$z = \max\left[(-x+t_{41}-t'_{21}),\ (t_{41}-t_{43}-t'_{21})\right] \tag{3-56}$$

2) There are several sets of equations for determining y and k, choose proper cases and use related equations. Use Fig. 3-16 for obtaining y and k.

CASE-1: $x+z \leq t_{12}$

$$t_{32} = \min\left[(x+y+z+k),\ (x+y+t'_{34})\right] \tag{3-57}$$

$$t_{42} = \min\left[(x+y+z+k),\ (z+k+t_{43})\right] \tag{3-58}$$

from which we get

$$y = \max\left[(-k+t_{32}-x-z),\ (t_{32}-x-t'_{34})\right] \tag{3-59}$$

$$k = \max\left[(-y+t_{42}-x-z),\ (t_{42}-z-t_{43})\right] \tag{3-60}$$

CASE-2: $x+z > t_{12}$

a. $x \geq t_{12}$

$$t_{32} = \min\left[(y+k+t_{12}),\ (y+t_{12}+t'_{34})\right] \tag{3-61}$$

$$y = \max\left[(-k+t_{32}-t_{12}),\ (t_{32}-t_{12}-t'_{34})\right] \tag{3-62}$$

b. $x < t_{12}$

$$t_{32} = \min\left[(y+k+t_{12}),\ (x+y+t'_{34})\right] \tag{3-63}$$

$$y = \max\left[(-k+t_{32}-t_{12}),\ (t_{32}-x-t'_{34})\right] \tag{3-64}$$

c. $z \geq t_{12}$

$$t_{42} = \min\left[(y+k+t_{12}),\ (k+t_{12}+t_{43})\right] \tag{3-65}$$

$$k = \max\left[(-y+t_{42}-t_{12}),\ (t_{42}-t_{12}-t_{43})\right] \tag{3-66}$$

d. $z < t$

$$t_{42} = \min\left[(y+k+t_{12}),\ (z+k+t_{43})\right] \tag{3-67}$$

$$k = \max\left[(-y+t_{42}-t_{12}),\ (t_{42}-z-t_{43})\right] \tag{3-68}$$

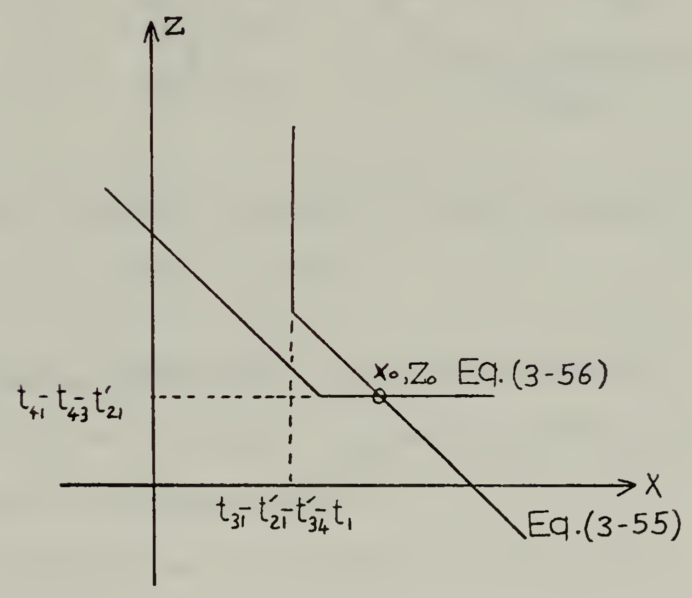



Fig. 3-15. "Curves" of (3-55) and (3-56)

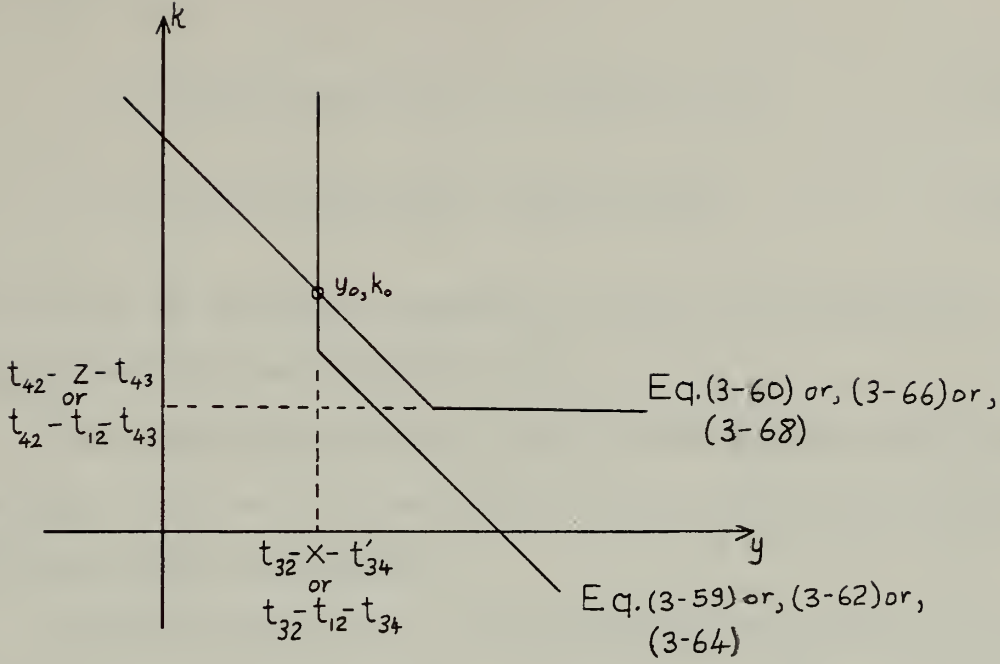



Fig. 3-16. "Curves" of (3-59) or (3-62) or (3-64) and (3-60) or (3-66) or (3-68)

Proof of Algorithm-B: Algorithm-B applies in the following cases, $t_{31} < t_{32}$ and $t_{42}$, $t_{41} < t_{32}$ and $t_{42}$. From equations (3-29) through (3-32) we can state that:

(a) x+y+z+k can not be min-cut for $t_{31}$ and $t_{41}$ because it is in $t_{32}$ and $t_{42}$.

(b) $x+y+t'_{34}$ and $z+k+t_{43}$ can not be min-cut for $t_{31}$ and $t_{41}$ respectively, because they are in $t_{32}$ and $t_{42}$ respectively.

Then the following two equations are obtained from (3-29) and (3-31) for $t_{31}$ and $t_{41}$:

$$t_{31} = \min\left[(x+z+t'_{21}),\ (x+t'_{21}+t'_{34}+t_1)\right] \tag{3-69}$$

$$t_{41} = \min\left[(x+z+t'_{21}),\ (z+t_{43}+t'_{21})\right] \tag{3-70}$$

from which we get

$$x = \max\left[(-z+t_{31}-t'_{21}),\ (t_{31}-t'_{21}-t'_{34}-t_1)\right] \tag{3-71}$$

$$z = \max\left[(-x+t_{41}-t'_{21}),\ (t_{41}-t_{43}-t'_{21})\right] \tag{3-72}$$

We can solve the above two equations for x and z or we may use a graphical solution for obtaining x and z.

For the determination of y and k, there are two cases to be considered: (1) $x+z \le t_{12}$ and (2) $x+z > t_{12}$. These cases will give us different sets of equations.

CASE-1. When $x+z \le t_{12}$, automatically $x \le t_{12}$ and $z \le t_{12}$. So $y+x+t'_{34} \le y+t_{12}+t'_{34}$ and $k+z+t_{43} \le k+t_{12}+t_{43}$, then $y+t_{12}+t'_{34}$ and $k+t_{12}+t_{43}$ are eliminated from (3-30) and (3-32) respectively. Also $x+y+z+k \le y+k+t_{12}$, then $y+k+t_{12}$ is eliminated from (3-30) and (3-32) because it can not be min-cut for $t_{32}$ and $t_{42}$. Thus, (3-30) and (3-32) become

$$t_{32} = \min\left[(x+y+z+k),\ (x+y+t'_{34})\right] \tag{3-73}$$

$$t_{42} = \min\left[(x+y+z+k),\ (z+k+t_{43})\right] \tag{3-74}$$

Which we obtain

$$y = \max\left[(-k+t_{32}-x-z),\ (t_{32}-x-t'_{34})\right] \tag{3-75}$$

$$k = \max\left[(-y+t_{42}-x-z),\ (t_{42}-z-t_{43})\right] \tag{3-76}$$

x and z are found before. In order to obtain y and k we can use a graphical solution using (3-75) and (3-76).

CASE-2. When $x+z > t_{12}$, we can eliminate $x+y+z+k$ from (3-30) and (3-32) since $x+y+z+k > y+k+t_{12}$ and it can not be min-cut for $t_{32}$ and $t_{42}$. In this case $x \lesseqgtr t_{12}$ and $z \lesseqgtr t_{12}$. Following the same reasoning given in the proof of Algorithm A, equations (3-30) and (3-32) become

$$t_{32} = \min\left[(y+k+t_{12}),\ (y+t_{12}+t'_{34}),\ (y+x+t'_{34})\right] \tag{3-77}$$

$$t_{42} = \min\left[(y+k+t_{12}),\ (k+t_{12}+t_{43}),\ (k+z+t_{43})\right] \tag{3-78}$$

$y+t_{12}+t'_{34}$ or $y+x+t'_{34}$ and $k+t_{12}+t_{43}$ or $k+z+t_{43}$ are eliminated from (3-77) and (3-78) respectively, and each case will give a set of equations for determining z and k.

Example 7. We shall realize the following terminal capacity matrix.

$$T = \left[\begin{array}{cc|cc} 1 & 5 & 3 & 3 \\ 4 & 2 & 3 & 3 \\ \hline 9 & 10 & 3 & 4 \\ 8 & 10 & 3 & 4 \end{array}\right] \tag{3-79}$$

The terminal capacity matrix is in Form-2, then realization can be done with Algorithm-B.

1) $$x = \max\left[(-z+8),\ (4)\right] \tag{3-80}$$

$$z = \max\left[(-x+7),\ (4)\right] \tag{3-81}$$

The solution of above equation is obtained from Fig. 3-17, x=4 and z=4.

2) $x+z=8 > t_{12}=5$, $x < t_{12}$ and $z < t_{12}$ then Case 2b and d. With the numerical values in (3-64) and (3-68), the following equations are obtained:

$$y = \max\left[(-k+5),\ (5)\right] \tag{3-82}$$

$$k = \max\left[(-y+5),\ (3)\right] \tag{3-83}$$

From Fig. 3-18, y=5 and k=3. Realization of the T matrix is given in Fig. 3-19.

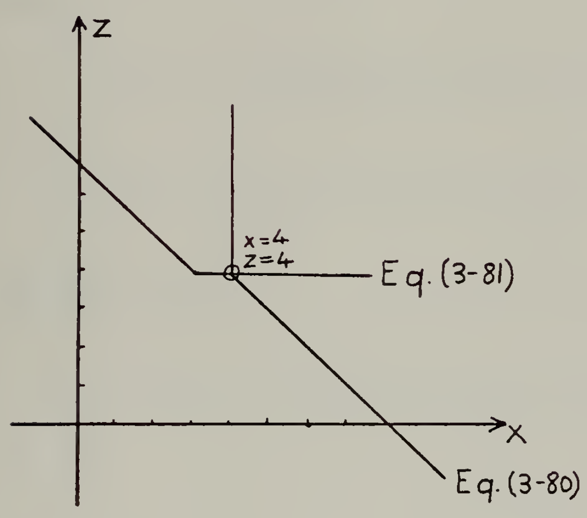



Fig. 3-17

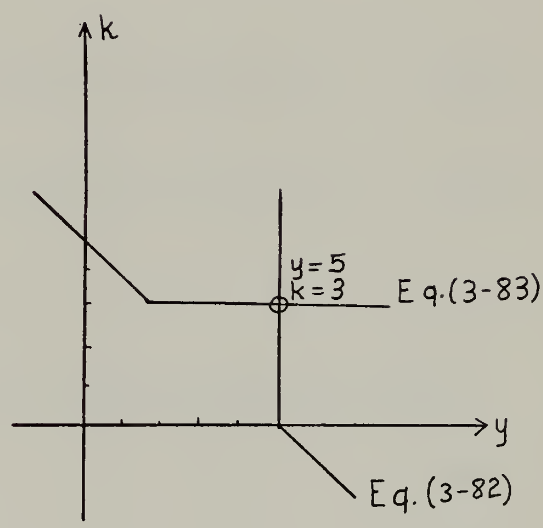



Fig. 3-18

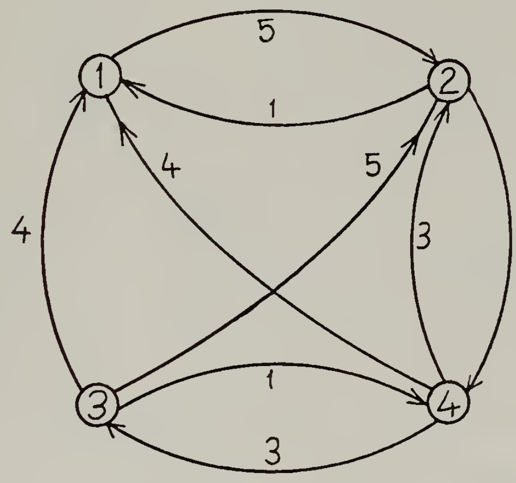



Fig. 3-19. Realization of T-matrix in (3-79)

3. Algorithm C

1) Determine x and k from the following equations.

$$x = t_{31}-t'_{21}-t'_{34}-t_1 \tag{3-84}$$

$$k = t_{42}-t_{43}-t_{12} \tag{3-85}$$

2) Determine y and z by writing the following equations with the aid of Fig. 3-20:

$$t_{32} = \min\left[(x+y+z+k),(y+k+t_{12}),(x+y+t'_{34}),(y+t_{12}+t'_{34})\right] \tag{3-86}$$

$$t_{41} = \min\left[(x+y+z+k),(z+k+t_{43}),(x+z+t'_{21}),(z+t_{43}+t'_{21})\right] \tag{3-87}$$

Which we get

$$y = \max\left[(-z+t_{32}-x-k),(t_{32}-k-t_{12}),(t_{32}-x-t'_{34}),(t_{32}-t_{12}-t'_{34})\right] \tag{3-88}$$

$$z = \max\left[(-y+t_{41}-x-k),(t_{41}-k-t_{43}),(t_{41}-x-t'_{21}),(t_{41}-t_{43}-t'_{21})\right] \tag{3-89}$$

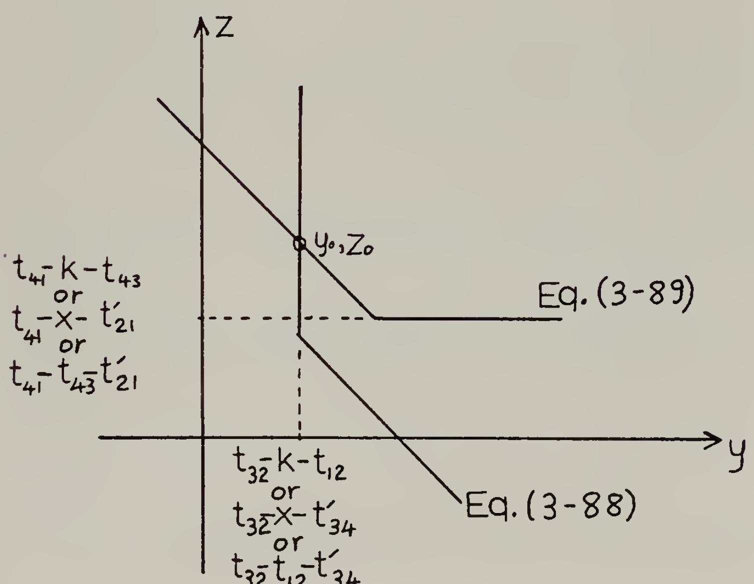



Fig. 3-20. "Curves" of (3-88) and (3-89)

Proof of Algorithm-C: Algorithm-C applies under the following cases, $t_{31} < t_{32}$ and $t_{41}$, $t_{42} < t_{32}$ and $t_{41}$. From equations (3-29) through (3-32) we can state that

(a) x+y+z+k can not be min-cut for $t_{31}$ and $t_{42}$ because it is in $t_{32}$ and $t_{41}$.

(b) $x+y+t'_{34}$ and $x+z+t'_{21}$ can not be min-cut for $t'_{31}$ because they are in $t_{32}$ and $t_{41}$ respectively.

(c) $y+k+t_{12}$ and $z+k+t_{43}$ can not be min-cut for $t_{42}$ because they are in $t_{32}$ and $t_{41}$ respectively.

Then the following two equations are obtained from (3-29) and (3-32).

$$t_{31} = x+t'_{21}+t'_{34}+t_1 \tag{3-90}$$

$$t_{42} = k+t_{43}+t_{12} \tag{3-91}$$

then

$$x = t_{31}-t'_{21}-t'_{34}-t_1 \tag{3-92}$$

$$k = t_{42}-t_{43}-t_{12} \tag{3-93}$$

For the determination of y and z, the following equations are used.

$$t_{32} = \min\left[(x+y+z+k),(y+k+t_{12}),(x+y+t'_{34}),(y+t_{12}+t'_{34})\right] \tag{3-94}$$

$$t_{41} = \min\left[(x+y+z+k),(z+k+t_{43}),(x+z+t'_{21}),(z+t_{43}+t'_{21})\right] \tag{3-95}$$

Which we obtain

$$y = \max\left[(-z+t_{32}-x-k),(t_{32}-k-t_{12}),(t_{32}-x-t'_{34}),(t_{32}-t_{12}-t'_{34})\right] \tag{3-96}$$

$$z = \max\left[(-y+t_{41}-x-k),(t_{41}-k-t_{43}),(t_{41}-x-t'_{21}),(t_{41}-t_{43}-t'_{21})\right] \tag{3-97}$$

x and k are obtained before, then in (3-96) we have three expressions ($t_{32}-k-t_{12}$, $t_{32}-x-t'_{34}$ and $t_{32}-t_{12}-t'_{34}$) which belong to the constant side of (3-96). Choosing the maximum of the above three expressions two are eliminated. Also in (3-97) choosing the maximum of the constant expressions two are eliminated. After the above simplifications two equations are obtained for determining y and z.

Example 8.: We shall realize the following terminal capacity matrix.

$$T = \left[\begin{array}{cc|cc} 1 & 3 & 2 & 2 \\ 5 & 2 & 2 & 2 \\ \hline 8 & 11 & 3 & 5 \\ 9 & 8 & 4 & 4 \end{array}\right] \tag{3-98}$$

The terminal capacity matrix is in Form-3, then the realization can be done using Algorithm-C.

1) x=0, k=1

2) $y = \max\left[(-z+10),(7),(8),(4)\right]$ (3-99)

$z = \max\left[(-y+8),(4),(6),(2)\right]$ (3-100)

which we can write

$$y = \max\left[(-z+10),(8)\right] \tag{3-101}$$

$$z = \max\left[(-y+8),(6)\right] \tag{3-102}$$

From Fig. 3-21 y=8 and z=6 are obtained. The realization of T matrix is given in Fig. 3-22.

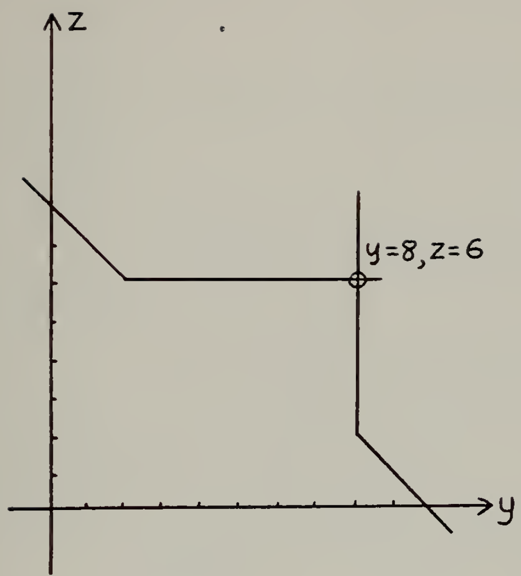



Fig. 3-21

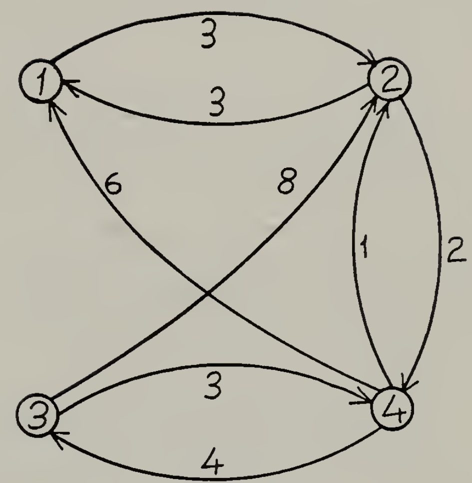



Fig. 3-22. Realization of T matrix

4. <u>Dominant Submatrix Partitioning of T Matrices</u>

The realization technique for the low-order case can be applied to the higher order T matrix if it can be partitioned, by rearranging the nodes, in the following manner:

a) Each submatrix corresponding to a sub-collection of nodes lying along the diagonal line is square.

b) The row of connection node in each diagonal submatrices contains elements with values no smaller than the value of any element in the column of T matrix after treating each diagonal xubmatrix as a node, where each column corresponds to a node which stands for diagonal submatrix. A connection node is a node in each diagonal submatrix which provides connection with the rest of the net.

c) The column of a connection node in each diagonal submatrix contains elements with values no smaller than the value of any element in the row of T matrix after treating each diagonal submatrix as a

node, where the row corresponds to a node which stands for diagonal submatrix.

These conditions are referred to as the "dominant conditions" of a T-matrix.

A T-matrix satisfying the dominant conditions is realizable if [1]:

1) Treating these submatrices along the diagonal line as nodes, the matrix T is realizable.

2) Each submatrix along the diagonal line is realizable.

Example 9: We shall realize the following terminal capacity matrix.

$$T = \left[\begin{array}{c:c:c:c}
\begin{matrix} 1 & 6 & 3 & 3 \\ 5 & 2 & 3 & 3 \\ 10 & 11 & 3 & 6 \\ 12 & 13 & 7 & 4 \end{matrix} & 1 & 1 & 1 \\ \hdashline
2 & \begin{matrix} 5 & 5 & 2 & 2 \\ 4 & 6 & 2 & 2 \\ 9 & 11 & 7 & 8 \\ 8 & 8 & 3 & 8 \end{matrix} & 1 & 1 \\ \hdashline
3 & 5 & \begin{matrix} 9 & 5 & 3 & 3 \\ 4 & 10 & 3 & 3 \\ 9 & 10 & 11 & 4 \\ 8 & 10 & 3 & 12 \end{matrix} & 2 \\ \hdashline
5 & 4 & 2 & \begin{matrix} 13 & 3 & 2 & 2 \\ 5 & 14 & 2 & 2 \\ 8 & 11 & 15 & 5 \\ 9 & 8 & 4 & 16 \end{matrix}
\end{array}\right] \qquad (3\text{-}103)$$

The T-matrix may be written as

$$T = \left[\begin{array}{cc:cc}
A & 1 & 1 & 1 \\
2 & B & 1 & 1 \\ \hdashline
3 & 5 & C & 2 \\
5 & 4 & 2 & D
\end{array}\right] \qquad (3\text{-}104)$$

A, B, C and D are the diagonal submatrices in the original T-matrix.

The realization of the T matrix is shown in Fig. 3-24, where A, B, C and D are the vertices. The realizations of A, B, C and D were done in Example 5 through 8 respectively. The final realization of the T-matrix is shown in Fig. 3-25.

The realization of (3-102) can be obtained using Algorithm-C.

1) $x=0$, $k=1$

2) $$y = \max\left[(-z+4),(4)\right] \tag{3-105}$$

$$z = \max\left[(-y+4),(4)\right] \tag{3-106}$$

From Fig. 3-23 $y=4$ and $z=4$ are obtained.

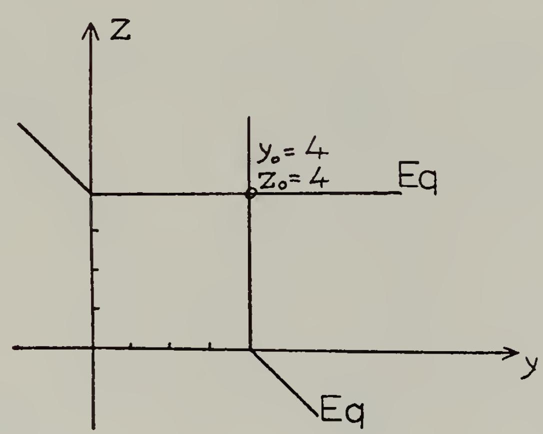



Fig. 3-23

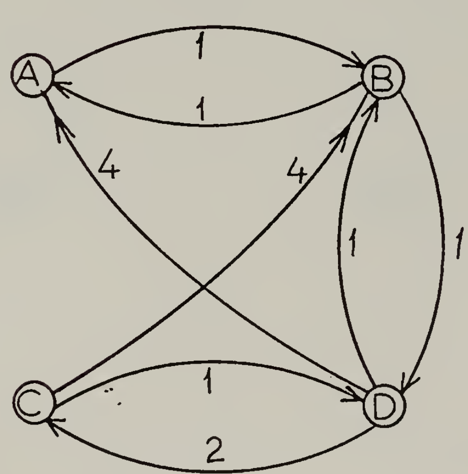



Fig. 3-24

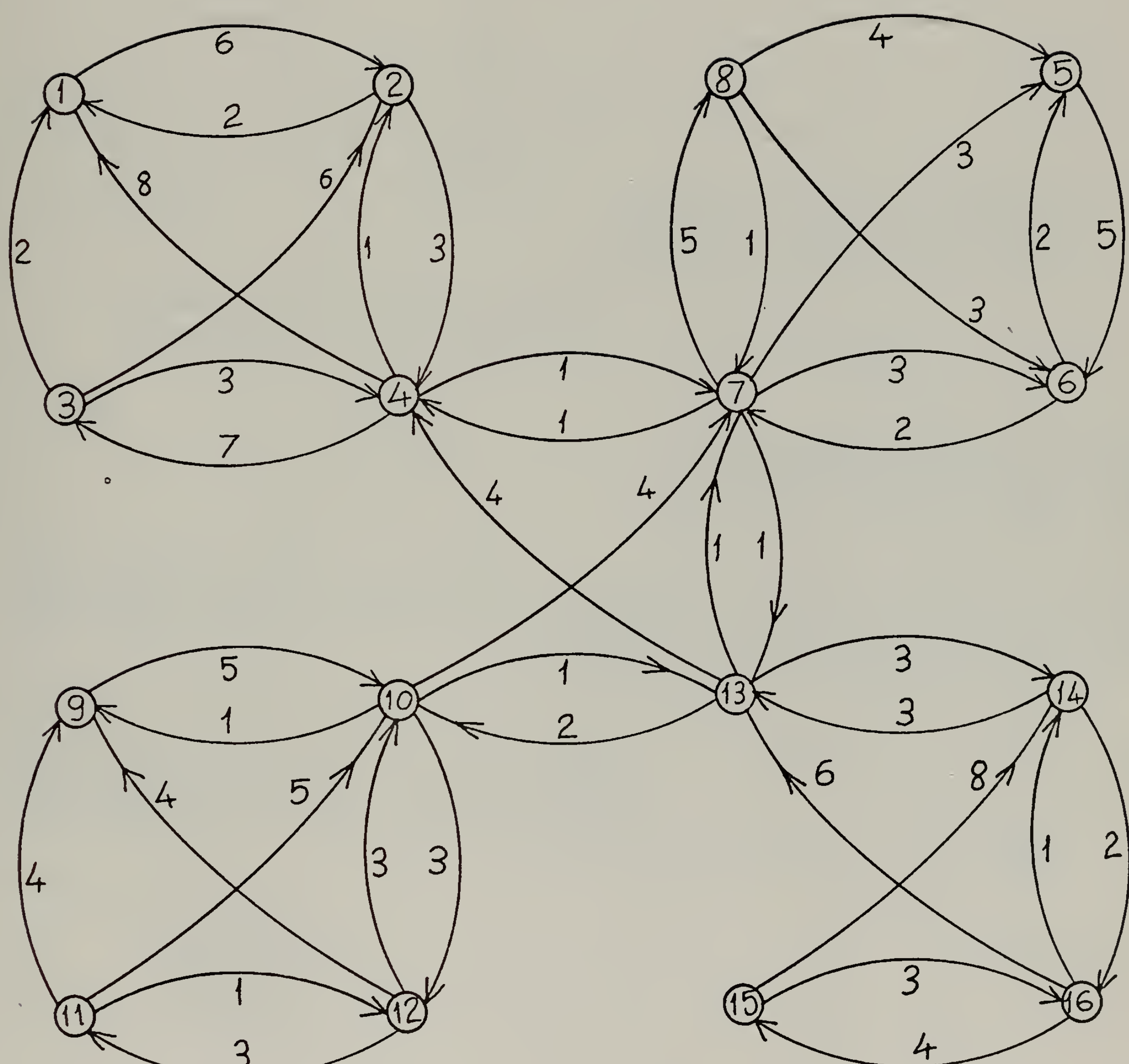
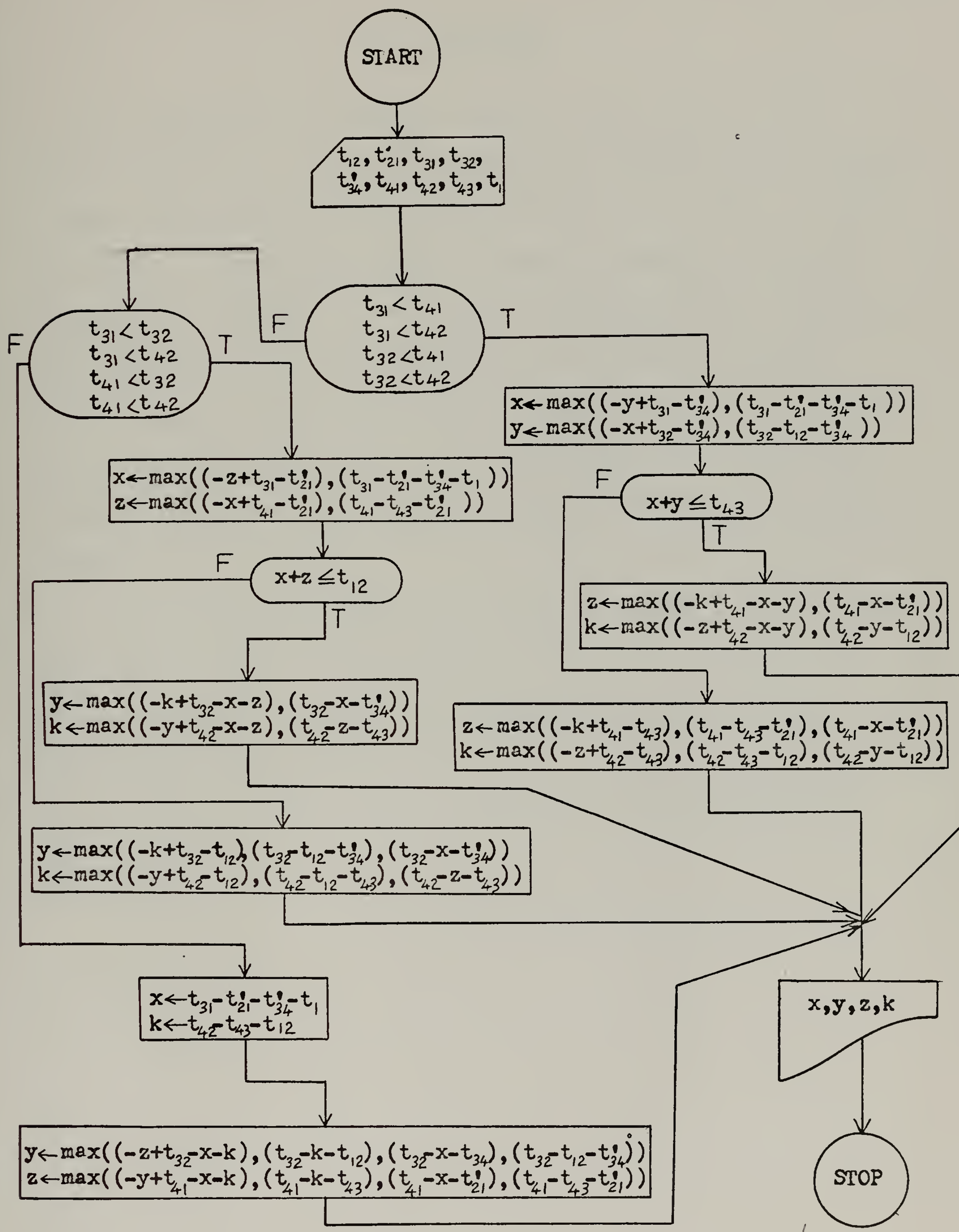


Fig. 3-25. Realization of T-matrix.

5. Flow Chart for Computer Programming



## IV. CONCLUSION

The realization of a terminal capacity matrix with oriented branch capacities is very complicated when the number of nodes becomes large. The techniques given in this paper will be useful for solving more complex communication system problems in practice. The given method can be extended to cover higher order cases without difficulty. The technique presented is effective for the following reasons: (a) The number of nodes in a subnet is increased compared to that in earlier methods. (b) Easy for hand calculation. (c) Adaptable to computer programing. The following problems are suggested for further studies:

1) To obtain a necessary and sufficient condition which is easy to check on a given terminal capacity matrix.

2) To adapt the realization techniques for the nonuniform cost function with minimum cost.

3) Write a computer program for the flow chart presented in this paper.

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3. REPORT TITLE

On Realization of Terminal Capacity Matrices

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Master's Thesis; December 1971

5. AUTHOR(S) *(First name, middle initial, last name)*

Tahsin Karan

| 6. REPORT DATE | 7a. TOTAL NO. OF PAGES | 7b. NO. OF REFS |
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13. ABSTRACT

This paper presents three algorithms for minimum cost synthesis of an oriented communication net. The realization technique is developed using the min-cut max-flow theorem. The algorithms are able to handle higher order terminal capacities compared to previous methods. Necessary and sufficient conditions are given for the application of the algorithms, which are suitable for computer implementation.

| KEY WORDS | LINK A | | LINK B | | LINK C | |
|---|---|---|---|---|---|---|
| | ROLE | WT | ROLE | WT | ROLE | WT |
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